



Newton-like method for singular 2-regular system of nonlinear equations

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Abstract

In this article the problem of solving a system of singular nonlinear equations will be discussed. The theory of local and Q-superlinear convergence for the nonlinear operators is developed.

1. Introduction

Let $F : D \subset R^n \rightarrow R^m$ be a nonlinear operator. The problem of solving a system of nonlinear equations consist in finding a solution $x^* \in D$ of the equation

$$F(x) = 0. \quad (1)$$

Definition 1

A linear operator $\Psi_2(h) : R^n \rightarrow R^m$, $h \in R^n$ is called 2-factoroperator, if

$$\Psi_2(h) = F'(x^*) + P^\perp F''(x^*)h, \quad (2)$$

where

P^\perp - denotes the orthogonal projection on $(\text{Im } F'(x))^{\perp}$ in R^n [1].

Definition 2

Operator F is called 2-regular in x^* on the element $h \in R^n$, $h \neq 0$, if the operator $\Psi_2(h)$ has the property:

$$\text{Im } \Psi_2(h) = R^m.$$

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Definition 3

Operator F is called 2-regular in x^* , if F is 2-regular on the set $K_2(x^*) \setminus \{0\}$, where

$$K_2(x^*) = \text{Ker}F'(x^*) \cap \text{Ker}^2 P^\perp F''(x^*), \quad (3)$$

$$\text{Ker}^2 P^\perp F''(x^*) = \{h \in R^n : P^\perp F''(x^*)[h]^2 = 0\}.$$

We need the following assumption on F :

A1) completely degenerated in x^* :

$$\text{Im} F'(x^*) = 0. \quad (4)$$

A2) operator F is 2-regular in x^* :

$$\text{Im} F''(x^*)h = R^m \text{ for } h \in K_2(x^*), h \neq 0. \quad (5)$$

A3)

$$\text{Ker}F''(x^*) \neq \{0\}. \quad (6)$$

If F satisfies A1 in x^* , then

$$K_2(x^*) = \text{Ker}^2 F''(x^*) = \{h \in R^n : F''(x^*)[h]^2 = 0\}. \quad (7)$$

In [1] it was proved, that if $n=m$, then the sequence

$$x_{k+1} = x_k - \left\{ \hat{F}'(x_k) + P_k^\perp F''(x_k)h_k \right\}^{-1} \cdot \left\{ F(x_k) + P_k^\perp F'(x_k)h_k \right\}, \quad (8)$$

where

P_k^\perp - denotes orthogonal projection on $(\text{Im} \hat{F}'(x_k))^\perp$ in R^n ,

$$h_k \in \text{Ker} \hat{F}'(x_k), \quad \|h_k\| = 1$$

converges Q-quadratically to x^* .

The matrices $\hat{F}'(x_k)$ obtained from $F'(x_k)$ by replacing all elements, whose absolute values do not increase $v > 0$, by zero, where $n = n_k = \|F(x_k)\|^{(1-a)/2}$, $0 < a < 1$.

In the case $n = m+1$ the operator

$$\left\{ \hat{F}'(x_k) + P_k^\perp F''(x_k)h_k \right\}^{-1}$$

in method (8) is replaced by the operator

$$\left[\hat{F}'(x_k) + P_k^\perp F''(x_k)h_k \right]^+ \quad (9)$$

and then the method converges Q-linearly to the set of solutions [2].

Under the assumptions A1-A3, the system of equation (1) is undetermined ($n > m$) and degenerated in x^* .

2. Extending of the system of equation

Now we construct the operator $\Phi : R^n \rightarrow R^{n-1}$ with the properties (4), (5) and such that $\Phi(x^*)=0$ [2].

Assume

A4) Let $F(x)=[f_1(x), f_2(x), \dots, f_m(x)]^T$, $n>m$ is two continuously differentiable in some neighbourhood $U \subset R^n$ of the point x^* .

Denote:

$$H = \text{lin}\{h\} \quad \text{for } h \in \text{Ker}^2 F'(x^*), h \neq 0.$$

$P = P_{H^\perp}$ denotes the orthogonal projection R^n on H^\perp

$$j_i(x) = P(f_i'(x))^T \quad \text{for } i=1,2,\dots,m.$$

For each system of indices $i_1, i_2, \dots, i_{n-m-1} \subset \{1, 2, \dots, m\}$ and vectors $h_1, h_2, \dots, h_{n-m-1} \subset R^n$ we define

$$\Phi(x) = \begin{bmatrix} F'(x)h \\ j(x) \end{bmatrix}, \tag{10}$$

where

$$j(x) : R^n \rightarrow R^r, \quad r=n-m-1,$$

$$j(x) = PF'(x)H^0, \quad H^0 = [h_1, h_2, \dots, h_r]^T,$$

$$j(x) = \begin{bmatrix} j_{i_1}(x)h_1 \\ \mathbf{M} \\ j_{i_r}(x)h_r \end{bmatrix}. \tag{11}$$

In [2] it was proved, that the sequence

$$x_{k+1} = x_k - [\Phi'(x_k)]^+ \cdot \Phi(x_k), \quad k=0,1,2,\dots \tag{12}$$

quadratically converges to the solution of (1).

3. New method

We propose the Newton-like method, where the sequence $\{x_k\}$ is defined by:

$$x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi(x_k). \tag{13}$$

The operator Φ' will by approximated by matrices $\{B_k\}$.

Let

$$s_k = x_{k+1} - x_k. \tag{14}$$

We propose matrices B_k which satisfy the secant equation:

$$B_{k+1}s_k = \Phi(x_{k+1}) - \Phi(x_k) \quad \text{for } k=0,1,2,\dots \tag{15}$$

For example, to obtain the sequence $\{B_k\}$ we can apply the Broyden method:

$$B_{k+1} = B_k - \frac{r_k s_k^T}{s_k^T s_k} \quad \text{for } k=0,1,2,\dots \quad (16)$$

where

$$r_k = \Phi(x_{k+1}) - \Phi(x_k) - B_k s_k. \quad (17)$$

We will prove for this method:

Q-linear convergence to x^* i.e. there exists $q \in (0,1)$ such, that

$$\|x_{k+1} - x^*\| \leq q \|x_k - x^*\| \quad \text{for } k = 0,1,2,\dots \quad (18)$$

and next *Q-superlinear convergence* to x^* , i.e.:

$$\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0. \quad (19)$$

We present the theorem which is an analogue of the Bounded Deterioration Theorem (Broyden, Dennis and More - [3]) for the Newton-like methods, when the operator $F'(x^*)$ is nonsingular.

Theorem 1 (The Bounded Deterioration Theorem)

Let F satisfies the assumptions A1-A4. If exist constants $q_1 \geq 0$ and $q_2 \geq 0$ such that matrices $\{B_k\}$ satisfy the inequality:

$$\|B_{k+1} - \Phi'(x^*)\| \leq (1 + q_1 r_k) \|B_k - \Phi'(x^*)\| + q_2 r_k, \quad (20)$$

then there are constants $e > 0$ i $d > 0$ such, that if

$$\|x_0 - x^*\| \leq e \quad \text{and} \quad \|B_0 - \Phi'(x^*)\| \leq d,$$

then the sequence

$$x_{k+1} = x_k - B_k^+ \Phi(x_k)$$

converges *Q-linearly* to x^* .

When the system of equation is rectangular, the proof of the theorem is analogous to that for the nonsingular and quadratic system and we neglect it.

Theorem 2 (Linear convergence)

Let F satisfies the assmptions A1-A4. Then the method

$$x_{k+1} = x_k - \{B_k\}^+ \cdot \Phi(x_k),$$

$$B_{k+1} = B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k}$$

locally and *Q-linearly* converges to x^* .

Proof.

To prove the Theorem we should prove the inequality (20) from Theorem 1. Now we notice:

$$\begin{aligned}
 \|B_{k+1} - \Phi'(x^*)\| &= \left\| B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k} - \Phi'(x^*) \right\| \leq \\
 &\leq \|B_k - \Phi'(x^*)\| + \left\| \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k} \right\| \leq \|B_k - \Phi'(x^*)\| + \\
 &+ \left\| \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - \Phi'(x^*) s_k + \Phi'(x^*) s_k - B_k s_k\} s_k^T}{s_k^T s_k} \right\| \leq \|B_k - \Phi'(x^*)\| + \\
 &+ \left\| \frac{(\Phi(x_{k+1}) - \Phi'(x^*)(x_{k+1} - x^*)) s_k^T}{s_k^T s_k} \right\| + \left\| \frac{(\Phi(x_k) - \Phi'(x^*)(x_k - x^*)) s_k^T}{s_k^T s_k} \right\| + \\
 &+ \left\| \frac{(\Phi'(x^*) - B_k) s_k s_k^T}{s_k^T s_k} \right\| \leq \|\Phi'(x^*) - B_k\| (1 + q_1 r_k) + c_1 \frac{\|x_{k+1} - x^*\|^2 \|s_k\|}{\|s_k^T s_k\|} + \\
 &+ c_2 \frac{\|x_k - x^*\|^2 \|s_k\|}{\|s_k^T s_k\|} \leq \|\Phi'(x^*) - B_k\| (1 + q_1 r_k) + q_2 r_k,
 \end{aligned}$$

where $c_1 > 0$, $c_2 > 0$, $q_1 > 0$, $q_2 > 0$, $r_k = \max\{\|x_{k+1} - x^*\|, \|x_k - x^*\|\}$. □

Theorem 3 (Q-superlinear convergence)

Let F satisfies the assumptions A1-A4 and the sequence

$$\begin{aligned}
 x_{k+1} &= x_k - \{B_k\}^{-1} \cdot \Phi(x_k), \\
 B_{k+1} &= B_k - \frac{\{\Phi(x_{k+1}) - \Phi(x_k) - B_k s_k\} s_k^T}{s_k^T s_k}
 \end{aligned}$$

linearly converges to x^* . Then the sequence $\{x_k\}$ Q-superlinearly converges to x^* .

Proof.

Matrices B_k satisfy secant equation (15), so

$$B_{k+1} = P_{L_k}^\perp B_k \tag{21}$$

where

$$L_k = \{X : X s_k = y_k, \text{ where } y_k = \Phi'(x_{k+1}) - \Phi'(x_k)\} \tag{22}$$

Denote

$$H_k = H(x_k, x_{k+1}) = \int_0^1 \Phi'(x_k + t(x_{k+1} - x_k)) dt.$$

We have $H_k \in L_k$ [4].

From (21) and [3] it follows:

$$\|B_{k+1} - B_k\|^2 + \|B_{k+1} - H_k\|^2 = \|B_k - H_k\|^2, \text{ for } i = 0, 1, 2, \dots$$

By lemma 2 [5] we get $\sum_{k=1}^{\infty} \|B_{k+1} - B_k\|^2 < \infty$, thus we obtain

$$\|B_{k+1} - B_k\| \rightarrow 0.$$

This denotes that the method (13)-(17) is Q-superlinearly convergent [6], which ends the proof. \square

4. Summary

The proposed method is Q-superlinearly convergent and easier to apply than the method (12), without calculation of $F''(x_k)$.

References

- [1] Izmailov A.F., Tretiakov A.A., *Factor-analysis of non-linear mappings*, Moscow, Nauka, Fizmatlit Publishing Company, (1994) 336.
- [2] Chmura A., Tretiakov A.A., *New Methods for the solution of nonlinear equations with singularities-Nonlinear Oscillations*, 2 (1999) 1.
- [3] Broyden C.G., J.E. Dennis, J.J. More, *On the local and superlinear convergence of quasi-Newton methods*, Journal Inst. Math. Appl., 12 (1973) 223.
- [4] Ortega J.M., Reinboldt W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).
- [5] Grzegórski S.M., *Rescaled multilevel least-change almost secant methods*, „Matematyka Stosowana” XXXIII, (1991).
- [6] Grzegórski S.M., *Orthogonal projections on convex sets for Newton-like methods*, SIAM J.Numer. Anal., 22 (1985) 1208.